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STRUCTURAL FATIGUE IN ONE-CRACK MODELS
WITH ARBITRARY INSPECTION

by

D. G. FORD

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SUMMARY

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Two FORTRAN programs have been developed from this formulation, one of which allows for random crack rates and run-time setting of inspections.

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16.   *ABSTRACT*

The reliability theory for fatigue lives of one-crack, two-stage models with hijacking has been extended to allow for arbitrary inspection intervals with possible arbitrary renewals. The life distribution is similar with a modified initiation density and the associated moment generating function is formally identical. It is possible to identify an unsteady Markov chain formed by the combined fatigue and inspection process.

Two FORTRAN programs have been developed from this formulation, one of which allows for random crack rates and run-time setting of inspections.

CONTENTS

$a\ (a)$	Crack size(s).
a_o	Initial crack size.
B	Infinite transition matrix, equation (4.12B).
$\mathbf{B_i} = [b_{ij}]$	Fifth order transition matrix for the inspection at time T_i.
$E.$	Expectation.
$f(a\|t)$	Conditional probability density of crack length (see $\phi(.)$, $0(.)$) at time t.
$f_a(t_a)$	Reliability type density of crack life (see Equation (2.6)).
$f_o(t), f_o(t_o)$	Density of initial life in the absence of any inspection or repair.
$f_*(t)$	Initial life density as affected by inspection or repair.
$f_{*i}(t)$	$f_*(t)$ defined in $(T_i, T_{i+1}]$ extended to (T_i, ∞).
$F.$	Cumulative density of distribution function, not necessarily proper.
$\bar{F}_a$	$1 - F_a$.
$\bar{F}_i$	$1 - F_{*i}$.
$g_i, g_i(t - T_i)$	Density of initial lives in structures repaired after inspection T_i. (This is initial data).
G_i	Distribution corresponding to g_i, not necessarily proper.
$h_*(t_o) = e^{-r_o t_o} f_*(t_o)$	Used for Equation (5.15A).
H_i	Defined by (5.10A). This is a hypothetical distribution of initial lives from a hijacked distribution for structures repaired at T_i.
I, J, K	Terms of (4.7); see (4.10).
$M.$	Generic form of moment generating function, $E. \exp(-u.)$.
M_j	Moment generating functions from g_j. See Section 4.4.
N_i	Defined by (5.15) and (5.15A).
$P, P_i(a) = P_i(t_a)$	Probability of rejection for an inspected structure with a crack of length a or at time t_a after initiation (operating characteristic). Normally used without subscripts.
$P_i = P(T_i - T_{io})$	Used in Section 4.3.
$\bar{P}_{io}$	Laplace transform of P_* with respect to t_a when $t_o = T_{io} +$ i.e. when the crack starts just after an inspection.
$P_j = P_j(T_j - t + t_a)$	The probability of rejection at an intermediate inspection (Section 3.2)
$\bar{P}_j = 1 - P_j\ Pr(.),$	
$Pr(.\|.)$	Probability, absolute or conditional.
$P_*(t_a\|t_o)$	Defined by (4.2). Overall acceptance probability during growth of some crack.

Q_i	Abbreviation used in I, Equation (4.8) and preceding.
$R = R(a) = R(t_a)$	Average local crack growth rate.
R_i	Probability of rejection at i-th inspection, time T_i. Equation (3.1).
$\mathbf{R} = \{R_i | i = 1, ..., \infty\}$	Infinite vector of R_i's.
$r(a) = r(t_a)$	Risk function.
$r_o = r(0)$	Hijack risk component.
$S(t)$	Arbitrary step function, Section 4.2.
$S(u)$	Laplace transform of $S(t)$.
t	Current equivalent time or cycles.
$t_.$	Other times according to subscript
T_i	Known inspection times, $T_o = 0$.
u, v, w	Laplace transform variables.
$\Delta T_i = T_{i+1} - T_i$	i-th inspection interval.
γ, δ	Real shifts for Bromwich inversion contours.
$\phi(t)$	Distribution of final lives without inspection, (2.6).
$\phi_*(t)$	Distribution of final lives with inspection and repair, (3.6).
$*$	Convolution function.

Subscripts

The subscripts associate the main symbol with the quantity listed.

o	Crack initiation without renewals.
$*$	Crack initiation affected by renewals.
$*_i$	As above but distinguished by inspection period.
a	Crack growth time.
c	Set complementation.
i	Inspection period $(T_i, T_{i+1}]$.
i_o, io	The inspection just before the start of the current crack i.e. $T_{io} < t_o \leqslant T_{io+1}$

Sets

Union and intersection are here denoted by "+" and the normal product convention while complements are indicated by the subscript c. Figures 1 and 5 illustrate some of the events.

A	Attrition.
E	Universe.
$M = CA_cR_c$	Mainstream — cracked but still in use.
R	Rejected at inspection.
ϕ	Empty set (see Section 5.1).

1. INTRODUCTION

It is a truism that fatigue life, especially with single cracks, consists of initiation time and crack growth time. The models based on this approach when the initiation is random, have been described in previous Reports[1,2,3]. In the last of these, considerations of continuity of probability, together with deterministic cracking, led to a first order partial differential equation (true for vector cracks also), which is the same as the continuity condition for compressible flow. It is also a degenerate form of the second order Fokker-Planck equation which it would become for random cracking. Most generally this describes the infinitesimal evolution of probabilities associated with continuous Markov processes[4,5].

In an earlier Report[3] the density of crack lengths (and thence that of total life) were found from the continuity equation without considering inspections, though these were mentioned briefly. Under the term hijacking it also introduced the effect of losses not due to fatigue. The present Report extends the previous solution to include inspections as well as the hijack risk. It considers the distribution of total life, the moment generating function, and the transition matrix for changes of state between inspections.

2. PREVIOUS RESULTS

Before proceeding, we shall summarise results for the one-crack model without inspection. As before, we shall use the generic notation f, F, ϕ, and M, to denote density, distribution functions and moment generating functions of their arguments. Density is affected by attrition and where necessary, these symbols will be subscripted to avoid confusion.

For vector cracks the continuity equation

$$\frac{Df}{Dt} = f(\operatorname{div} \mathbf{R}(\mathbf{a}) + r(\mathbf{a})) \tag{2.1}$$

holds, where D denotes total derivative and $\mathbf{R}(\mathbf{a}) = d\mathbf{a}/dt$, the previously averaged crack rate, $r(\mathbf{a}) = $ total risk function including hijack risk, and
div $\mathbf{R}(\mathbf{a}) = dR/da$ for a single cracking, a known function of crack length.

When this is expanded, one obtains the degenerate Fokker-Planck equation (for single cracks)

$$\frac{\partial f}{\partial t} + R(a)\frac{\partial f}{\partial a} = -f\left(\frac{d}{da}R(a) + r(a)\right) \tag{2.2}$$

with the characteristic equations

$$dt = da/R(a) = -df/f(dR/da + r(a)). \tag{2.3}$$

In this the crack trajectories are characteristics and the general solution for crack length density is

$$f = f(a_o|t_o)\frac{R(a_o)}{R(a)} \exp -\int_{a_o}^{a} \frac{r(a)}{R(a)}\,da$$

$$= f(a_o|t_o)\frac{R(a_o)}{R(a)} \exp -\int_{0}^{t_a} r(t)\,dt \tag{2.4}$$

introducing the growth time t_a to reach crack length a. Here a_o is the initial crack size, a constant, and t_o, the initial life, corresponds to the crack $a|t$. The growth time

1

$$t_a = t - t_o.$$

When the hijack risks are denoted r_o, and the boundary conditions are included $r(t) = r(\alpha(t))$

$$f(a|t) = \frac{e^{-r_o t_o} f_o(t_o)}{R(a)} \exp\left(-\int_0^{t_a} r(t)dt\right) \tag{2.5}$$

where the generic f refers to different density functions according to its subscript.

It is now possible to average the risk $r(a)$ at time t for the overall life density.

$$\phi(t) = r_o e^{-r_o t_o} [1 - F_o(t)] + e^{-r_o t_o} f_o(t_o) * f_a(t_a) \tag{2.6}$$

where $f_a(t_a) = r(t_a) \exp - \int_0^{t_a} r(t)dt$, the reliability based crack life density.

The moment generating function, $E_\phi \exp -ut$, follows as

$$M_\phi(u) = M_o(u + r_o) M_a(u) + \frac{r_o}{u + r_o}\left[1 - M_o(u + r_o)\right] \tag{2.7}$$

which differs slightly from the corresponding equation of Reference 3 since $-u$ is used here to assure convergence for positive u.

2.1 Effect of Inspections

This is twofold: in the first place, the density of crack length will be reduced suddenly at a number of steps corresponding to each inspection. Secondly, structures rejected at an inspection may be returned, after repair, to the population. This brings the whole problem into a close relationship with statistical renewal theory. However, the latter does not include hijack risks, nor the two stages involved in fatigue.

We will first consider renewals and then the corresponding moment generating functions. In the following, it is most convenient to have all variables as time, and continue to use t_o, t_a and t to denote initiation time, crack period and their sum. The subscript $*$ will refer to densities or generating functions affected by renewals. Inspection times are T_i, $i = 0, 1, 2$ etc. where $T_o = 0$. The i-th inspection interval and the associated quantities occur after T_i.
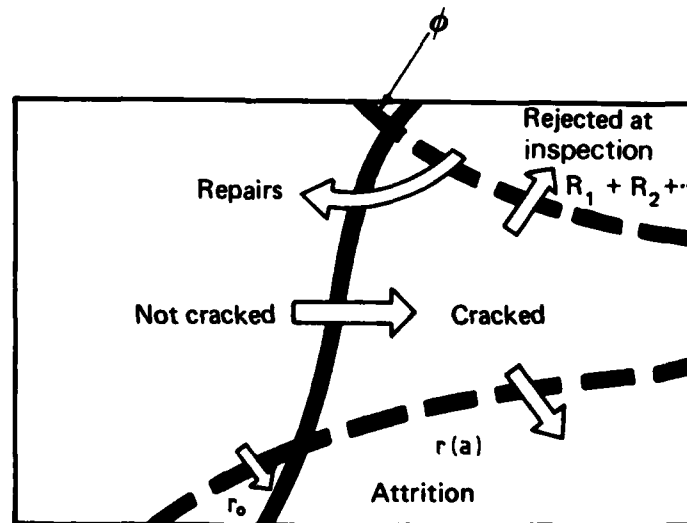

FIG 1. POSSIBLE STATES OF STRUCTURES

2

3. LIFE DISTRIBUTION

At a particular time the population of structures suffering two-stage fatigue with inspections and attrition may be described by the Venn diagram of Figure 1. In the course of time the arrows indicate the evolution of members of the population; one may equivalently imagine uniform measure density over the rectangle and the subset boundaries moving left and inwards opposite to the arrows. The zero set corresponds to rejection of uncracked structures. This is the general process we intend to describe.

3.1 Renewal Allowance for Single Cracks

At T_i — just before inspection, suppose the partial density of vector crack lengths is $f(\mathbf{a}|T_i-)$, which therefore becomes

$$f(\mathbf{a}|T_i+) = (1 - P_i(\mathbf{a})) f(\mathbf{a}|T_i-)$$

just afterwards. With respect to the whole population this gives the probability of rejection at the i-th inspection as

$$R_i = \int_0^\infty P_i(\mathbf{a}) f(\mathbf{a}|T_i-) \, d\mathbf{a} \tag{3.1}$$

We also define P_i the operating characteristic of the inspection method, in terms of growth time so that $P_i(a) = P_i(t_a)$. The context will indicate which definition is current.

After T_i, the population will also include repaired or modified structures returned from inspection. It may also be depleted by those retired from service. Statistically, these repairs or retirement affect the initial life distribution. After each inspection we will assume that structures are repaired to the same condition (not necessarily the original as new condition) despite the possibility of differing crack lengths being discovered. (Since it is mathematically easy, the restored condition shall be assumed specific to the particular inspection).

This assumption requires us to first consider the density of crack initiation which may determine boundary conditions for $f(a|t > T_i)$ (see Ref. 3).

For one crack in the absence of inspections, let $e^{-r_0 t}(1 - F_o(t))$ be the survivorship function for initial life when hijack attrition is included (Ref. 3). When inspections are included but not, for the moment, hijacking, let the corresponding survivorship fraction (based on the original population) be $F_{*i-1}(t)$ for $t < T_i$ say.

Just after an inspection all the uncracked structures will be retained and also augmented by the fraction $R_i[1 - G_i(0)]$ of inspected and repaired structures.
Symbolically

$$1 - F_{*i}(T_i+) = [1 - F_{*i-1}(T_i-)] + R_i[1 - G_i(0)] \tag{3.2}$$

where $G_i(t)$ is a general, possibly partial, distribution function of initiation at times $T_i + t$ of structures repaired at time T_i. At time t, $T_i < t < T_{i+1}$ with *hijacking* try

$$e^{-r_0 t}(1 - F_{*i}(t)) = e^{-r_0 t}(1 - F_{*i-1}(t)) + R_i e^{-r_0(t-T_i)}(1 - G_i(t - T_i))$$

or

$$\bar{F}_i = \bar{F}_{i-1}(t) + H_i \tag{3.3}$$

where $\bar{F} = 1 - F$ and

$$H_i = e^{r_0 T_i} R_i[1 - G_i(t - T_i)]$$

Recursive substitution then leads to

$$\bar{F}_i(t) = \bar{F}_o + \sum_{j=1}^i H_j \tag{3.3A}$$

which requires an interpretation of F_{*o}, G_o etc.

3

Initially (3.3) indicates that

$$1 - F_{*0}(T_o) = 1 - F_{*-1}(T_o) + e^{r_o T_o} R_o[1 - G_o(0)] \tag{3.3B}$$

and we would expect that

$$F_{*0}(t \mid t < T_t) = F_o(t),$$

the initial failure distribution. If we define $T_o = 0$ then for $0 < t < T_1$, (3.3) becomes

$$1 - F_{*0}(t) = 1 - F_o(t) = 1 - F_{*-1}(t) + R_o[1 - G_o(t - T_o)] \tag{3.3C}$$

The definitions may now be arbitrary. The most convenient for us is $F_{*-1}(t) \equiv F_o(t)$ and $R_o = 0$ leaving G_o arbitrary.

(Note that all F_{*i}, G_i etc. are defined on $(0, \infty)$; F_{*i} is applied on (T_i, T_{i+1}) and $F_* \equiv F_{*i}$ etc. everywhere with the index suited to t.) To summarise T_o, $R_o = 0$, $F_{*-1}(t) = F_{*0}(t) = F_o(t)$ and G_o is arbitrary.

In (3.3) or (3.3A) the restored factor $\exp{-r_o t}$ allows for hijacking. These equations now provide some of the boundary conditions for $f(a \mid T_i < t < T_{i+1})$ when the solution has marched to $T_{i+1} - $. Then (3.1) provides R_{i+1}, allowing (3.3C) to operate over the next inspection interval (T_{i+1}, T_{i+2}).

3.2 Density Function for Attrition

Now include the hijack factor in (3.3C) and consider the density of initiation

$$-\frac{d}{dt} e^{-r_o t} \bar{F}_i = r_o e^{-r_o t} \bar{F}_i + e^{-r_o t} f_{*i}(t) \tag{3.4A}$$

where the first term is obviously the local incidence of hijacking and therefore a direct component of the density $\phi(t)$ of total life.

From (3.3A) the second term is

$$e^{-r_o t} \sum_{j=1}^{i} e^{r_o T_j} R_j g_j(t - T_j) \tag{3.4B}$$

where g_j is the initiation density corresponding to G_j which may be partial or defective.

Consider $f(a \mid T_i < t < T_{i+1})$ after the i-th inspection. Such a crack would have begun at $t - t_a(a)$ and would therefore have been inspected an integral number of times. The operating characteristic $P(a)$ of any crack is the defined in terms of crack length. In keeping with our present approach, it is more convenient to define this in terms of the growth time t_a as $P(t_a)$. We further define P_j as the characteristic corresponding to the j-th inspection of a crack.
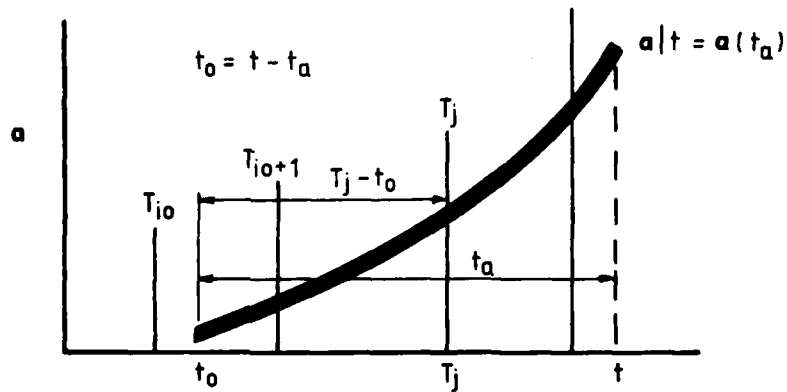

FIG 2. NOTATION FOR INSPECTION DURING CRACK GROWTH

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In the notation of the figure

$$P_j = P(T_j - t + t_a).$$

For a long crack, several non-trivial P_j's are possible. Unless specifically stated, several P_j's in the same expression refer to the same crack trajectory.

Now consider the basic density $f(a|t)$ allowing attrition of cracked structures and also inspections. If $T_i < t < T_{i+1}$ integration (Ref. 3) along the characteristic and multiplications by $\bar{P}_j = 1 - P$ produce

$$R(a)f(a|t) = \bar{P}_{io+1} \ldots \bar{P}_i \exp\left(-\int_{t_o}^{t} r(a)dt\right) e^{-r_o t_o} f_{*io}(t - t_a)$$

$$= \left(\prod_{io+1}^{i} \bar{P}_j\right) \exp\left(-\int_{0}^{t_a} r(t)dt\right) e^{-r_o t_o} \sum_{j=o}^{i_o} r_o T_j R_j g_j(t_o - T_j) \qquad (3.5)$$

where t_a is the growth time and $T_{io} < t - t_a < T_{io+1}$. As before, with the hijacking, this produces the attrition density

$$\Phi_*(t|T_i < t < T_{i+1}) = r_o e^{-r_o t} \bar{F}_i + \int_{a_o}^{\infty} r(a)f(a|t)da$$

$$= r_o e^{-r_o t}[1 - F_{*t}(t)] + \int_{0}^{t} e^{-r_o t_o} f_*(t_o) \prod_{i_o+1}^{i} \bar{P}(T_j - t_o) dF_a(t_a) \qquad (3.6)$$

with $dF_a(t_a)$ from (2.6), the initiation $t_o = t - t_a$, as always, and $\bar{P}_j$ is described in full. In (3.5) the finite upper limit t is no restriction because $f_* = 0$ for negative arguments. Recognition of this will aid the manipulations below.

4. MOMENT GENERATING FUNCTIONS

Equation (3.6) appears to be a convolution but the status of $\bar{P}_j$ is uncertain.
Let us form the moment generating function as

$$\sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \phi(t|i) e^{-ut}dt = \int_{0}^{\infty} r_o e^{-(r_o+u)t} [1 - F_*(t)]dt +$$

$$\sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \int_{0}^{t} e^{-(r_o+u)t_o - ut_a} f_*(t_o) \prod_{i_o+1}^{i} \bar{P}(T_j - t_o) dF_a(t_a)dt, \qquad (4.1)$$

where f_* is the global form of f_{*t}, the whole of the modified initial life distribution.

4.1 Inspection Functions

Consider the product of the inspection factors as functions of initiation t_o and of growth time t_a. For fixed t_o this term is a step function of t_a (determining $i - i_o$) but it is at least piecewise continuous in t_o since, almost always, $T_j - t_o$ continuously determines crack sizes at inspection.

Occasionally, through i_o decreasing, t_o introduces another factor, but if $P_j(a_o) = 0$, which we now assume, continuity with respect to t_o will be retained.

Let us abbreviate the inspection factor

$$\prod_{i_o+1}^{i} (1 - P(T_j - t_o)) = P_*(t_a|t_o) \equiv P_* \qquad (4.2)$$

where P_* is a step function decreasing from unity. We now intend to treat (4.1) as a convolution by taking some of the inspection terms as part of the crack life distribution. In (4.1) the region of integration is the infinite sector shown in Figure 3. The operation began with integration

along strips such as AB which were then extended to horizontal inspection bands whose contributions were finally summed.
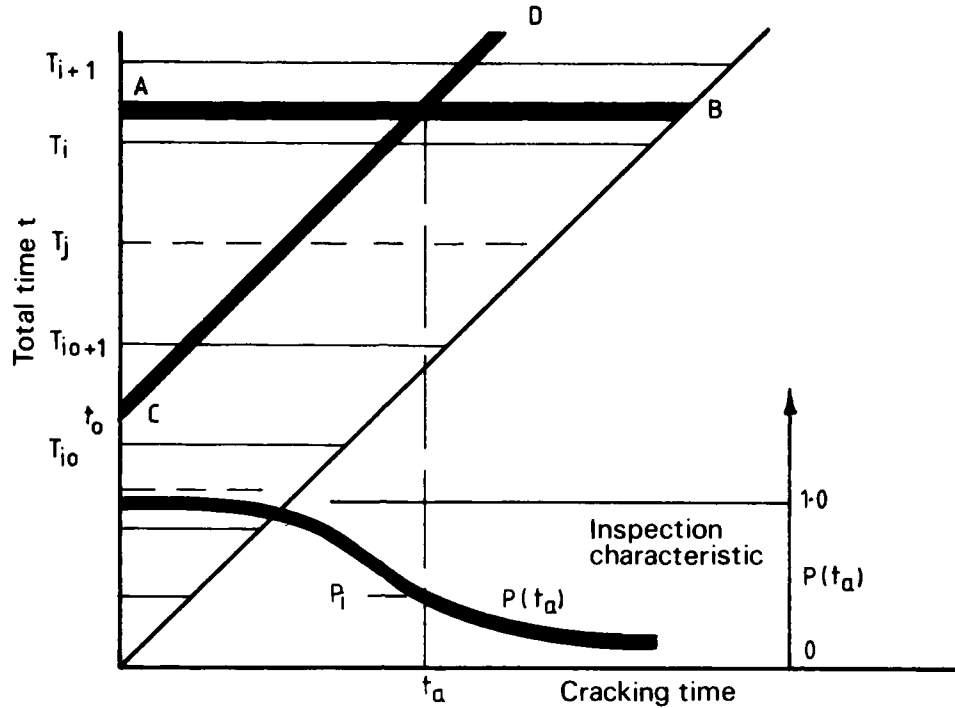

FIG. 3 REGION OF INTEGRATION FOR MOMENT GENERATING FUNCTION

If we change the variable t to t_o, then we may do the first integration along CD and include the summation by adopting the limits of $(0, \infty)$ for t_a.

Thequus with (4.2) ation (4.1) becomes

$$M_\phi(u) = \frac{r_o}{u + r_o} (1 - M_\ast(u + r_o))$$

$$+ \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \int_0^{\infty} f_\ast(t_o)e^{-(u+r_o)t_o} P_\ast e^{-ut_a} dF_a(t_a)dt_o \qquad (4.3)$$

where $M_\ast = E_\ast \exp(-ut_o) = MGF$ of $f_\ast$.

When $P_\ast \equiv 1$ the integral reduces to the uninspected form $M_\ast(u + r_o)M_a(u)$; in general, these terms introduce convolutions of transforms.

4.2 Step Function Transforms

We have already seen that $P_\ast$ is a step function for given t_o and in the figure for (4.1) the first integral along CD traverses an infinite number of inspections. In (4.3) then $P_\ast$, given by (4.2) is interpreted for all t_a. Thus $P_\ast(t_a|t_o)f_a(t_a)$ may be regarded as a defective conditional density of cracking life.

By the convolution theorem for Laplace transforms

$$\int_0^{\infty} P_\ast(t_a)f_a(t_a)e^{-ut_a} dt_a = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{P}_\ast(u - v)M_a(u)dv \qquad (4.4)$$

where the bar indicates a Laplace transform, as is M_a.

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For any step function $S(t)$ such as that below
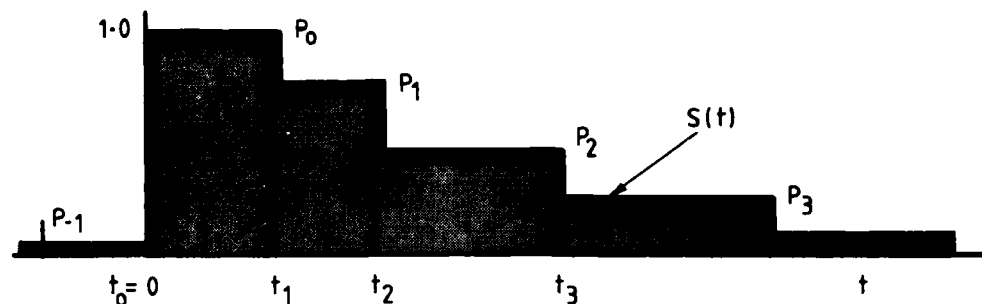
FIG 4. TYPICAL STEP FUNCTION

it is easily shown that, in the notation indicated,

$$S(u) = u^{-1} \sum_{i=0}^{\infty} (P_i - P_{i-1})e^{-ut_i} \qquad (4.5)$$

In the present instance with steps from (4.2), one would have

$$t_1 = T_{io+1} - t_o, \quad \dots \quad , t_j = T_{io+j} - t_o,$$

with t_o arbitrary. Part of the t_o dependence is in the value of $T_{io+1} - t_o$.

Let $\bar{P}_{io}$ be the transform of P_* corresponding to $t_o = T_{io} + \epsilon$. Then for t_o elsewhere in this inspection interval, the transform of the step function is

$$e^{u(t_o - T_{io})}\bar{P}_{io} + u^{-1}(1 - e^{u(t_o - T_{io})}) \qquad (4.6)$$

The first term has a simple shift operator but the second "spillage" term is required because the backwards shift truncates $S(t)$ for $t < 0$.

4.3 Complete Moment Generating Function

We now perform the transforms indicated by (4.3) requiring the transform of step functions. From (4.5)

$$\bar{P}_{io} = u^{-1} \sum_{i_o}^{\infty} (P_i - P_{i-1})e^{-u(T_i - T_{io})}, \quad P_t = 1, \quad P_{io} = 1, \quad P_{io-1} = 0,$$

giving (4.6) as

$$e^{ut_o} u^{-1} \sum_{i_o}^{\infty} (P_i - P_{i-1})e^{-uT_o} + u^{-1}(1 - e^{ut_o - uT_{io}})$$

with
$$P_i = P(T_i - T_{io}).$$

The inner integral (4.4) of (4.3) is now

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v^{-1}e^{vt_o} \sum_{i_o}^{\infty} (P_i - P_{i-1})e^{-vT_i} M_a(u - v)dv$$

$$+ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{u - v} M_a(v)dv - \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v^{-1}e^{v(T_o - T_{io})}M_a(u - v)dv \quad (4.7)$$

with the $u - v$ argument variously placed.

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All the components of (4.7) are similar so that we now substitute the first into (4.3) to obtain

$$I = \sum_{i_o=0}^{\infty} \int_{T_{io}}^{T_{io+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_*(t_o)e^{-(u-v+r_o)t_o}M_a(u-v)Q_{to}\,dvdt_o$$

where $Q_{to} = \sum_{i_o}^{\infty} (P_i - P_{i-1})e^{-vT_i}/v$.

If the order of integration is reversed then in each interval the inspection factor is constant with respect to t_o. Thus the t_o limits of I may be infinite provided the inspection factor is regarded as a step function. The t_o integral is thus the transform of the products of f_* and this step function. This introduces another convolution whence

$$I = \frac{1}{(2\pi i)^2} \int\int M_a(u-v)\,M_*(u-v+r_o) \sum_{i=0}^{\infty} (Q_i - Q_{i-1}) \frac{e^{-(u-v-w+r_o)T_i}}{u-v-w+r_o}\,dvdw$$

where $Q_{-1} = 0$.

In the w-plane there is a pole at $u-v+r_o$. Contour integration then leaves the cancelling residues,

$$I = \frac{1}{2\pi i} \int M_a(u-v)M_*(u-v+r_o) \sum_{i=0}^{\infty} (Q_i - Q_{i-1}) \frac{dv}{v} = 0 \qquad (4.8)$$

The next component of (4.7) leads to

$$J = \sum_{i_o=0}^{\infty} \int_{T_{io}}^{T_{io+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_*(t_o)e^{-u+r_o)t_o}M_a(u-v) \frac{dv}{v}\,dt_o$$

$$= \frac{1}{2\pi i} M_o(u+r_o) \int_{\gamma-i\infty}^{\gamma+i\infty} M_a(u-v) \frac{dv}{v} \text{, absorbing the summation,}$$

$$= M_*(u+r_o)\overline{H(t)f_a(t)} = M_*(u+r_o)M_a(u) \qquad (4.9)$$

using the convolution theorem.

In the third component of (4.7) the presence of T_{io} indicates the presence of another step function. In (4.3) it leads to

$$K = -\sum_{i_o=0}^{\infty} \int_{T_{io}}^{T_{io+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_*(t_o) \frac{e^{-(u-v+r_o)t_o}}{v} M_a(u-v)e^{-vT_{io}}\,dv$$

where the step function heights are $v^{-1} \exp -vT_{to}$. In the same way as before, using (4.5),

$$K = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} M_a(u-v)M_*(u-v+r_o) \sum_{i=0}^{\infty} \frac{e^{-(v-w)T_{io}} - e^{-(v-w)T_{io-1}}}{v-w}\,dvdw$$

Once again the only w-pole is for $w = v$ in which case the residues again cancel; $K = 0$.

When all these results are assembled ((4.3), 4.8) and (4.9))

$$M_\phi(u) = \text{Hijack term} + I + J + K$$

$$= \frac{r_o}{u+r_o}\left[1 - M_o(u+r_o)\right] + M_*(u+r_o)M_a(u) \qquad (4.10)$$

which is exactly the same as it would be without inspections except that $M_* = f_*$ and includes renewals. This last fact means in general that $M_\phi(0) \neq 1$. Other cumulants of the total life are shown in Table 1.

TABLE 1

Cumulants of Fatigue Life in Terms of Initiation and Crack Time Moments

$$M_\bullet(u) = \text{MGF of cycles to initiation}$$

$$M_\bullet^{(k)}(-r_o) = \mu_k + O(r_o) \quad \text{where } \mu_k = k\text{th moment}, \ \mu_o \equiv 1 - \sum_{k=1}^{\infty} R_k[1 - G_k(\infty)]$$

$$M_o(u) = 1 + \alpha_1 u + \tfrac{1}{2}\alpha_2 u^2 \ldots, \qquad\qquad \alpha_o \equiv 1,$$

$$= \text{MGF of cracking time, moments } \alpha_k.$$

Put

$$m_k \equiv M_\bullet^{(k)}(r_o); \qquad A_k \equiv \alpha_k - k\alpha_{k-1}/r_o,$$

where

$$r_o \equiv \text{Risk of loads above ultimate and of hijacking.}$$

Then for total life we have the cumulants:

$$\kappa_1 = m_o A_1 + r_o^{-1}; \qquad\qquad\qquad\qquad\qquad \text{(Mean)}$$

$$\kappa_2 = 2m_1 A_1 + m_o A_2 - m_o^2 A_1^2 + r_o^{-2} \qquad\qquad\qquad \text{(Variance)}$$

$$\kappa_3 = 3m_2 A_1 + 3m_1 A_2 + m_o A_3 - 6m_o m_1 A_1^2 - 3m_o^2 A_1 A_2 + 2m_o^3 A_1^3 + 2/r_o^3$$

$$\kappa_4 = 4m_3 A_1 + 6m_2 A_2 + 4m_1 A_3 + m_o A_4 - 12m_o m_2 A_1^2 - 24m_o m_1 A_1 A_2 + 24m_o^2 m_1 A_1^3$$

$$- 12m_1^2 A_1^2 - m_o^2\{4A_1 A_3 + 3A_2^2\} + 12m_o^3 A_1^2 A_2 - 6m_o^4 A_1^4 + 6/r_o^4$$

4.4 Transform of f.

Though informative, equation (4.10) is still not presented in terms of basic MGFs, relying as it has on $M_\bullet$. To relate $M_\bullet$ to M_o and other known MGFs define $f_\bullet$ piecewise from (3.4). Then its transform

$$M_\bullet(u) = \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} e^{-ut} f_{\bullet i}(t)\,dt$$

$$= M_o(u) + \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \sum_{j=1}^{i} e^{r_o T_j - ut} R_j g_j(t - T_j)\,dt$$

in which $M_o(u) = E_o \exp - u t_o$. The R_j, the overall probabilities of rejection at times T_j, are effectively constants.

Changing the order of summation absorbs one sum to produce

$$M_\bullet(u) = M_o(u) + \sum_{j=1}^{\infty} R_j e^{r_o T_j} \int_{T_j}^{\infty} e^{-ut} g_j(t - T_j)\,dt$$

$$= \sum_{j=0}^{\infty} R_j\, e^{-(u-r_o)T_j}\, M_j(u), \quad R_o = 1 \text{ here,}$$

defining $M_j(u) = \text{MGF of } g_j(t)$.

From (3.1) and (3.4)

$$R_j = \int_{a_0}^{\infty} P_{j_0-1} \ldots P_{j-1} P_j e^{-r_0 t_0} \left[1 - F_a(t_a)\right] f_{*j_0}(t_0) \frac{da}{R(a)}$$

$$= \int_0^{T_j} \prod_{j_0+1}^{j-1} P_k P_j(t_a) e^{-r_0 t_0} \bar{F}_a(t_a)/_{*j_0}(t_0) dt_a$$

where $\bar{F}_a(t_a) = 1 - F_a(t_a)$ and the finite upper limit is set by the fact that initiation $t_0 > 0$. This expands into

$$R_j = \int_0^{T_j} \bar{P}(T_{j_0+1} - t_0) \ldots \bar{P}(T_{j-1} - t_0)P(t_a)e^{-r_0 t_0}[1 - F_a(t_a)] f_{*j_0}(t_0) \, dt_a \qquad (4.11)$$

where the factors $\bar{P}$, P depend implicitly on t_a also and

$$t_0 = T_j - t_a.$$

Now we know that

$$f_{*j}(t) = \sum_{k=0}^{j} e^{-r_0 T_k} R_k g_k(t - T_k), \qquad g_0 \equiv f_0(t), \qquad (4.11A)$$

whence

$$R_j = \int_0^{T_j} \bar{P}(T_{j_0+1} - T_j + t_a) \ldots \bar{P}(T_{j-1} - T_j + t_a)\bar{F}(t_a)P_a(t_a)$$

$$\times \sum_{k=0}^{j_0} e^{-r_0 T_k} R_k g_k(T_j - T_k + t_a) \, dt_a \qquad (4.12)$$

This is essentially an infinite set of recursive linear equations for R_j ($R_0 = 1$). If their matrix form is

$$\mathbf{R} = \mathbf{BR} \qquad \mathbf{B} = [b_{ij}]$$

then $b_{ij} = \int_0^{T_j} \bar{P}(T_{j_0+1} - T_j + t_a) \ldots \bar{P}(T_{j-1} - T_j + t_a)P(t_a)\bar{F}_a(t_a)$

$$\times e^{-r_0 T_i} g_i(T_j - T_i + t_a)dt_a \qquad \text{if } i \leqslant j; \ t_a, T_j \to j_0,$$

$$= 0 \qquad \qquad \text{if } i > j, \qquad (4.12B)$$

all obtainable in principle by simple quadratures. Since R_0 is unity these equations are not homogeneous.

5. EMBEDDED MARKOV PROCESS

In another interpretation the b_{ij} are transition probabilities (rejection and repair) in a transient discrete Markov process embedded in the attrition and fatigue process. The element b_{ij} represents probabilities of structures repaired after time T_i being rejected again at the inspection T_j.

However, in the equation $\mathbf{R} = \mathbf{BR}$, the matrix $\mathbf{B}$ consists of transition probabilities among an infinite number of states. Nevertheless, these states are still incomplete, not including attrition, hijacking or even new cracking. Furthermore, the infinite vector $\mathbf{R}$ is described by a single transition $\mathbf{B}$ covering all time. It is more convenient to consider transitions just after each inspection time T_i; the states at such times may be regarded as agglomerations of the states represented by b_{ij}. Thus, structures cracked at T_i include rejections and repairs from previous inspections.

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In this way, the process **R** may be identified with a series of transitions of a smaller process after each inspection. From the Venn diagram of Figure 1, the number of states in this smaller process is six-three for cracked and three for uncracked structures. One of these states has zero probability (although it may be considered an absorbing state) so that a 5×5 transition matrix is required for each inspection time. These will be denoted **B** for time T_i. The state ignored is the rejection of uncracked structures at inspection.

To find **B**$_i$, it is necessary to consider the possible combinations of various states. This amounts to gathering previous results and formalising assumptions, implicit or otherwise. We consider the epochs $T_i +$ just after any inspection and imagine the "mainstream" of structures as having survived the initial hijacking at rate r_o, becoming cracked by time T_i, and *then* being subdivided into the mutually exclusive states of "immediate" (i.e. at $T_i +$) rejection, attrition in the i-th interval $(T_i, T_{i+1}]$ and the mainstream for T_{i+1}. The last will be augmented, as at each inspection, by repaired structures cracking again; one of the state transitions to be considered.

5.1 State Probabilities

We begin by considering the state probabilities at $T_i +$ which may also be normalisers for the conditional transition probabilities. Comparison with $T_{i+1} +$ and the use of known results then provide elements of **B**. Let $\phi(t)$ be the integrated attrition and abbreviate $\phi(T_i)$, $F_s(T_i)$ to ϕ and F_{si}. In addition C, A and R are sets of respectively cracked, failed or hijacked and rejected structures; before time $T_i +$ in the first instance. Set complements are subscripted c and unions and intersections are most conveniently denoted by $+$ and the product convention.

Among the eight factors of $(C + C_c)(A + A_c)(R + R_c)$ three are empty, namely:

CAR Inspection and rejection of a cracked structure *after* attrition;

C_cAR Inspection and rejection for hijacked structures; and

C_cA_cR Rejection of ordinary uncracked structures.

After obvious condensations this leaves the universe

$$E = CR + CA + CA_cR_c + C_cA + C_cA_c \tag{5.1}$$

in which C, A and their complements describe the end result of past history but R, R_c refer to the one inspection, beginning the interval $(T_i, T_{i+1}]$.

At time $T_i +$ we know that

$$Pr(CR) = R_i \text{ from (3.3) with } t = T_i \tag{5.2}$$

The basic hijack allowance states that the mainstream C_cA_c of uncracked structures is $\exp(-r_oT_i)(1 - F_{si})$.

Without hijacking, $Pr(C_cA_c)$ would reduce to $1 - F_{si}$ so that the difference

$$Pr(C_cA) = (1 - e^{-r_oT_i})(1 - F_{si}), \tag{5.3}$$

the fraction for hijacked but uncracked structures.

Hence for those already cracked

$$Pr(CR + CA + CA_cR_c) = F_{si}, \tag{5.4}$$

the cracked mainstream.

By definition $\Phi_i = Pr(A) = Pr(C_cA + CA)$

from which subtraction provides

$$Pr(CA) = \Phi_i - (1 - e^{-r_oT_i})[1 - F_{si}] \tag{5.5}$$

Now

$$Pr \text{ (cracked mainstream)} = Pr(CA + CA_cR_c + CR)$$

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and by subtraction

$$Pr(CA_cR_c) = F_{ai} - R_i + (1 - e^{-r_0T_i})[1 - F_{ai}] - \Phi_i = 1 - e^{-r_0T_i}[1 - F_{ai}] - R_i - \Phi_i \quad (5.6)$$

This is *not* the mainstream at T_{i-1} since it will be affected by new cracking and attrition during the interval $(T_i, T_{i+1}]$.
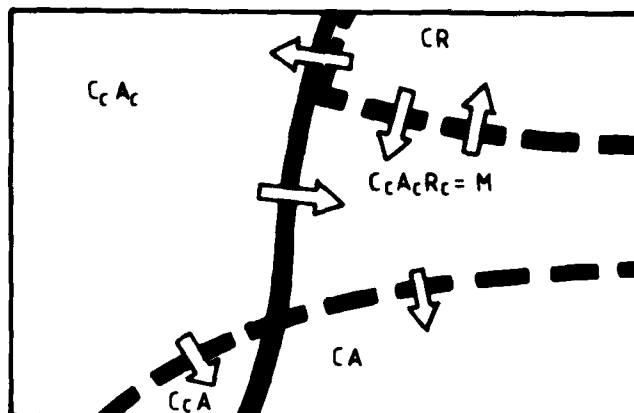
FIG 5. STATES AFTER EACH INSPECTION

5.2 Transition Probabilities

Equations (5.1) to (5.6) describe the five states shown in Figure 5 and their probabilities. Now consider transitions between them during $(T_i, T_{i+1}]$. The main difference from the previously used infinitesimal Markov process is that the finite inspection interval allows transition from repaired or uncracked structures to any state.

Obviously attrition is always an absorbing state. The process as a whole is therefore transient and the ultimate fate of all structures is either retirement, or attrition by failure or hijacking. We shall now examine transitions out of the five basic conditions using set theory and probability where necessary.

The more difficult elements will require interpretation of various terms in the expressions (3.5), (3.6) or (3.3) for attrition or rejection. Integrals of these may appear as convolution functions.

5.2.1 CR

In terms of time variates this has the total measure

$$R_i = \int_0^{T_i} \prod_{i_o+1}^{i-1} \beta_k P(t_a) e^{-r_0t_0}[1 - F_a(t_a)]f_a(t_0)dt_a, \quad t_0 = T_i - t_a.$$

which leads to the component of initiation density $R_i e^{-r_0(t_0-T_i)}g_i(t_0-T_i)$.

The arguments of Section 3 apply equally to all components so that in the interval $(T_i T_{i+1}]$ this corresponds to the partial density of crack length

$$(R_i/R(a))e^{-r_0(t_0-T_i)}[1 - F_a(t_a)]g_i(t - T_i - t_a), \quad t_0 = t - t_a, \quad (5.7)$$

which forms part of (3.5).

Corresponding to (3.6) one finds the attrition rate

$$\phi(t|\text{From } R_i) = R_i r_0 e^{-r_0(t-T_i)}[1 - G_i(t - T_i)]$$

$$+ R_i \int_0^{t - T_i} e^{-r_0(t_0-T_i)}g_i(t - T_i - t_a)dF_a(t_a)$$

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from which integration shows

$$R_i Pr(CR_i \to C_cA|R_i) = R_i \int_{T_i}^{T_{i+1}} r_o e^{-r_o(t-T_i)}[1 - G_i(t - T_i)]dt \qquad (5.8)$$

and

$$R_i Pr(CR_i \to CA|R_i) = R_i \int_{T_i}^{T_{i+1}} \int_0^{t-T_i} e^{-r_o(t_o-T_i)} g_i(t - T_i - t_a) \, dF_a(t_a)dt \qquad (5.9)$$

The first of these may be integrated by repeating the general argument for (5.3). The transition element is therefore

$$Pr(CR_i \to C_cA|R_i) = (1 - e^{-r_o\Delta T_i})(1 - G_i(\Delta T_i)), \qquad (5.10)$$

where $\Delta T_i = T_{i+1} - T_i$.

Equation (5.9) has no simpler form but it may be interpreted as a probability. Since

$$1 - H_i(t) = e^{-r_o t}[1 - G_i(t)] \qquad (5.10A)$$

is a survival probability under attrition, its complement is a true distribution function and never defective. Then, as a convolution,

$$Pr(CR_i \to A|R_i) = Pr(t_o + t_a < \Delta T_i)$$

$$= H_i * F_a(\Delta T_i), \text{ say,}$$

where $t_o \sim H_i(t_o)$, H_i from above, and $t_a \sim F_a(t_a)$
for the reliability based crack growth time. This includes $R_i \to C_iA$ but the time does not allow inspection. By subtraction of (5.10)

$$Pr(CR_i \to CA|R_i) = H_i * F_a(\Delta T_i) - (1 - e^{-r_o\Delta T_i})(1 - G_i(\Delta T_i)), \qquad (5.11)$$

we have already noted that

$$Pr(CR_i \to C_cA_c|R_i) = e^{-r_o\Delta T_i}(1 - G_i(\Delta T_i)) \qquad (5.12)$$

augmenting the mainstream of uncracked structures.

There remain two more transitions to CR_{i+1} or CA_cR_c, in the cracked mainstream. The first of these follows in the same way as R_i itself, but from the partial density (5.7). Thus

$$Pr(CR_i \to CR|R_i) = \int_0^{\Delta T_i} e^{-r_o(t_o-T_i)} P(t_a)[1 - F_a(t_a)]g_i(\Delta T_i - t_a)dt_a$$

If we recall our assumption $P(0) = 0$, this takes the form

$$\int_0^{\Delta T_i} P(t_a)[1 - F_a(t_a)]dH_i(\Delta T_i - t_a) = (P[1 - F_a]) * H_i(\Delta T_i) \qquad (5.13)$$

Finally, for transitions to the cracked mainstream,

$$Pr(CR_i \to CA_cR_c|R_i) = 1 - \text{Eq. (5.10)} - \ldots - (5.13)$$

$$= H_i(\Delta T_i) - H_i * ((1 - P)F_a + P)(\Delta T_i) \qquad (5.14)$$

$$= ((1 - P)[1 - F_a]) * H_i(\Delta T_i).$$

5.2.2 $CA_cR_c = M$

This is defined by (5.6) for entry to the inspection interval $(T_i, T_{i+1}]$. Just after T_{i+1}, it has changed by rejection and an influx of structures repaired after the previous inspection T_i.

Obviously $M \to C_cA$ and $M \to C_cA_c$ are impossible; also the influx of cracked structures, newly repaired and otherwise, complicates $Pr(M_i \to M_{i+1}|M_i)$.

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Equation (5.6) provides the unconditional $Pr(M_{i+1})$ as well as $Pr(M_i)$ but this influx makes it more convenient to start again from the crack length density (3.4). At time T_{i+1}, the only cracks from M_i must be those for which $t_a > \Delta T_i$. If (3.4) is restricted to these at $t = T_{i+1}$, then all attrition during $(T_i, T_{i+1}]$ is accounted for, and also the rejections R_{i+1}. Then, changing the cracked length variate to t_a,

$$Pr(M_i)Pr(M_{i+1}|M_i) = \int_{\Delta T_i}^{T_{i+1}} \left(\prod_{i_o+1}^{i+1} P_j\right)[1 - F_a(t_a)]e^{-r_o t_o} \sum_{j=o}^{t_o} R_j g_j(t_o - T_j)dt a$$

with $t_o = T_{i+1} - t_a$. This differs from $Pr(M_{i+1})$ only by

$$N_i = \int_0^{\Delta T_i} (1 - P(t_a))[1 - Fa(t_a)]e^{-r_o t_o} f^*(T_{i+1} - t_a)dt_a \qquad (5.15)$$

after some obvious simplifications. As in (5.11) it would now be possible to define a distribution $1 - (\exp - r_o t)(1 - F_a(t))$, and thus define this term as another convolution. However, this includes elements of $C_c A$, the attrition of uncracked structures, and to abbreviate N_i we use the defective density $h_a(t_o) = (\exp - r_o t_o)f_a(t_o)$ so that

$$N_i = ((1 - P)[1 - F_a])^* h_a(\Delta T_i) \qquad (5.15A)$$

If we are guided by context we may conveniently let $Pr(M_i) = M_i$, using the same symbols for the event and its probability. Then, using (5.6) and (5.15),

$$Pr(M_i \to M_{i+1}|M_i) = (M_{i+1} - N_i)/M_i = \frac{1 - e^{-r_o T_{i+1}}[1 - F_{ai+1}] - R_{i+1} - \Phi_{i+1} - N_i}{1 - e^{-r_o T_i}[1 - F_{ai}] - R_i - \Phi_i} \qquad (5.16)$$

The difference $M_i + N_i - M_{i+1}$ must now be divided among $M_i \to CR_{i+1}$ and $M_i \to CA_{i+1}$ to find the corresponding transition probabilities. Consider the attrition of cracked mainstream structures during $(T_i, T_{i+1}]$ given by (3.6), and based on (3.5). In the latter, CA_cR_c or M_i correspond to use of the partial density $f_{ai-1}(t_o)$ during the i-th inspection interval. This excludes attrition of structures repaired at T_i. If they are cracked however, their absolute probability from (5.11) is known to be

$$R_i H_i^* F_a(\Delta T_i) - (5.10) = R_i Pr(CR_i \to CA|R_i).$$

In (3.6), the second term is the attrition density of previously cracked structures, and those repaired at T_i. Writing the second term as

$$\phi_* - r_o(\exp - r_o t)[1 - F_a(t)]$$

and allowing for (5.16)

$$M_i Pr(CA_cR_c \to CA|T_i < t < T_{i+1}) = \int_{T_i}^{T_{i+1}} (\phi_a(t) - r_o e^{-r_o t}[1 - F_a(t)])dt +$$

$$+ R_i(1 - e^{-r_o \Delta T_i})[1 - G_i(\Delta T_i)] - R_i H_i^* F_a(\Delta T_i)$$

The second term of the integral follows as before from the increment of (5.4). After some reduction this becomes

$$M_i Pr(M \to CA|i\text{-th}) = \Delta\Phi_i + \Delta F_{ai}(1 - e^{-r_o T_{i+1}}) - R_i H_i^* F_a(\Delta T_i)$$

$$- (1 - e^{-r_o \Delta T_i})\{e^{-r_o T_i}[1 - F_{ai}] - R_i[1 - G_i(\Delta T_i)]\} \qquad (5.17)$$

From the remarks below (5.15)

$$M_i Pr(M_i \to CR_{i+1}|i\text{-th}) = M_i + N_i - M_{i+1} - \text{Eq.}(5.17)$$

which eventually reduces to $\qquad M_i Pr(M_i \to CR_{i+1}|i\text{-th}) =$

$$R_{i+1} - R_i - \Delta F_{ai} + N_i + R_i\{H_i^* F_a(\Delta T_i) - (1 - e^{-r_o T_i})[1 - G_i(\Delta T_i)]\}$$

with R_i, N_i and H_i defined by (3.1) or (4.11) - (4.12), (5.15) and (5.10A).

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5.2.3 C_cA_c

This is the only remaining non-trivial state. Any transition is possible from it and its absolute probability at $T_i +$ is

$$Pr(C_cA_c) \quad e^{-r_0 T_i}[1 - F_{si}]$$

During $(T_i, T_{i+1}]$ $Pr(C_cA)$ increases by the increment of (5.3) and this comes from $C_cA_c + CR_i$. Thus,

$$e^{-r_0 T_i}[1 - F_{si}]Pr(C_cA_c \to C_cA|i\text{-th}) = \Delta Pr(C_cA) - R_i Pr(CR_i \to C_cA|R_i)$$

$$= (1 - e^{-r_0 \Delta T_i})\{e^{-r_0 T_i}[1 - F_{si}] - R_i[1 - G_i(\Delta T_i)]\} - \Delta F_{si}(1 - e^{-r_0 T_{i+1}}) \qquad (5.19)$$

using (5.3) and (5.10).

As a whole, C_i comes from $C_cA_c + CR_i$ and of course its increment has measure ΔF_{si}, the decrease in C_c. For the increment from C_cA_c, subtract the transitions from CR_i giving

$$Pr(C_cA_c \to C_{i+1}) = \Delta F_{si} - R_i Pr(R_i \to C(A + R + M)|R_i)$$

$$= \Delta F_{si} - R_i\{H_i(\Delta T_i) - (1 - e^{-r_0 \Delta T_i})[1 - G_i(\Delta T_i)]\} \qquad (5.20)$$

which follows from (5.11), (5.13) and (5.14) respectively.

Consider

$$- \Delta Pr(C_cA_c) = Pr(C_cA_c)\{Pr(C_cA_c \to C_{i+1}|i\text{-th}) + Pr(C_cA_c \to C_c(A + A_c)|i\text{-th}\}$$

whence by transposing (5.19) and (5.20)

$$e^{-r_0 T_i}[1 - F_{si}]Pr(C_cA_c \to C_cA_c|i\text{-th}) = R_i H_i(\Delta T_i) - \Delta F_{si}e^{-r_0 \Delta T_i}(1 - e^{-r_0 T_i}) \qquad (5.21)$$

with H_i from (5.10A).

As was done with transitions $CR_i \to C_{i+1}$ we must now partition (5.20). After subtracting the correction term in (5.19) for $R_i \to C_{i+1}$, one may imagine the three types of transition during $(T_i, T_{i+1}]$ being driven by f_{si-1}. Attrition is simplest with a contribution from part of (3.6) minus (5.11). Thus, the appropriate rate is

$$\phi(t:T_i < t < T_{i+1}) = \int_0^{t-T_i} e^{-r_0(t-t_a)}f_{si-1}(t - t_a)\, dF_a(t_a)$$

$$= \int_0^{t-T_i} e^{-r_0 t_0}(f_s(t_0) - R_i g_i(t_0 - T_i))\, dF_a(t_a)$$

Over the inspection interval this integrates to

$$e^{-r_0 T_i}[1 - F_{si}]Pr(C_cA_c \to CA|i\text{-th})$$

$$= \Delta \Phi_i - e^{-r_0 T_i}(1 - e^{-r_0 \Delta T_i})[1 - F_{si}] - R_i\{H_i{}^* F_a(\Delta T_i) - (1 - e^{r_0 \Delta T_i})[1 - G_i(\Delta T_i)]\} \qquad (5.22)$$

using (5.11) for the subtracted term.

Inspection of (3.5) for $t = T_{i+1} +$ indicates the crack length density

$$R(a)f(a|T_{i+1}) = e^{-r_0 t_0}[1 - F_a(t_a)](1 - P(t_a))f_{si-1}(t_0)$$

for structures from C_cA_c. For rejected structures, we replace $1 - P$ by its complement and integrate to obtain

$$e^{-r_0 T_i}[1 - F_{si}]Pr(C_cA_c \to CR_{i+1}|i\text{-th}) = (P(1 - F_a))^* H_d(\Delta T_i) - R_i(P[1 - F_a])^* H_i(\Delta T_i) \qquad (5.23)$$

using (5.13) with the assumption $P(O) = 0$ and (5.15A), (5.10A) for definitions of H_* and H_i.

We now know probabilities for transition to $CA + CR_{i+1}$ and to C_{i+1} as a whole from equations (5.22), (5.23) and (5.20). Since the transition events are disjoint

$$e^{-r_0 T_i}[1 - F_{si}]Pr(C_cA_c \to M_{i+1}|i\text{-th}) = \Delta F_{si} - \Delta \Phi_i + e^{-r_0 T_i}(1 - e^{-r_0 \Delta T_i})[1 - F_i]$$

$$- R_i N_i - (P[1 - F_a])^* H_d(\Delta T_i) \qquad (5.24)$$

using (5.15A).

5.3 Transition Matrices

It is now possible to arrange the elements above into transition matrices for each epoch T_i+. In many cases, normalising to conditional probabilities introduce awkward fractions so that it is most convenient to place the normalisers in a diagonal prefactor. Doing this, and gathering elements from above, leads to the matrices shown in Tables 2 and 3.

Let $\mathbf{P}^{-1} = [P_i]^{-1}$ be the prefactor where only P_4 and P_5 differ from unity.

From Section 5.1 we know the absolute state probabilities of Table 2 which provide P_4 and P_5.

<div align="center">

TABLE 2
Normalising Factors

</div>

j	State	Name	Absolute Probability	P_j
1	C_cA	Hijacked	$(1 - e^{-r_0 T_i})[1 - F_{*i}]$	1
2	CA	Failed or hijacked	$\Phi_i - (1 - e^{-r_0 T_i})[1 - F_{*i}]$	1
3	CR	Rejected	R_i	1
4	$CA_cR_c = M$	Cracked mainstream	$1 - R_i - \Phi_i - e^{-r_0 T_i}[1 - F_{*i}]$	*
5	C_cA_c	Uncracked mainstream	$e^{-r_0 T_i}[1 - F_{*i}]$	*

* As in previous column.

In previous derivations, all results depend on functions which are fully defined at time T_i+ on the range of crack lengths $0 \cup [a_0, \infty)$ which imply the other functions needed from $(T, T_{i+1}]$. This validates the Markovian nature of the discrete transitions.

6. CONCLUSIONS

Using the same procedures, the previous analysis[3] for fatigue life distributions in one-crack models with hijacking have been extended to cases with arbitrary renewals at general inspection intervals.

The conclusions are similar to those derived previously if one postulates a defective distribution f_* of crack initiation lives as affected by renewals, and includes factors depending on the operating characteristic of the inspections. The moment generating function has the same form as before[3] and with $f_0(t_0)$ replaced by $f_*(t_0)$. The proof of this is facilitated by treating inspection and crack life density together.

The two-stage fatigue process is Markovian in continuous time. At any time a structure must be in one of five states corresponding to combinations of attrition, cracking and rejection at inspection or their complements. Transition between these states is a variable Markov chain embedded in the continuous process with epochs conveniently placed just after inspections.

6.1 Implementation

Life distributions, risk rates and rejection probabilities may be found from two FORTRAN IV programs developed from the preceding theory. The second of these has options for random crack rates and/or run-time setting of inspections. This generalisation and the program will be described in two further reports.

TABLE 3
Transition Matrix

From \ To	-1	C_cA	CA	CR	$CA_cR_c = M$	C_cA_c
C_cA	1			0	0	0
CA	1			0	0	0
CR	1	$(1-\rho)G_t$	$H_t*F_a(\Delta T_t) - (1-\rho)G_t$	$(\bar P F_a)*H_t(\Delta T_t)$	$(\bar P F_a)*H_t(\Delta T_t) = N_t$	ρG_t
$CA_cR_c = M$	$1 - \tau_t F_t$ $- R_t - \Phi_t$	0	$\Delta F_{et}(1-\tau_{t+1}) + \Delta\Phi_t$ $- R_tH_t*F_a(\Delta T_t) - \tau_t(1-\rho)G_t$ $+ R_t(1-\rho)F_t$	$R_{t+1} - R_t - \Delta F_{et}$ $+ R_tH_t*F_a(\Delta T_t)$ $- R_t(1-\rho)G_t + N_t$	$1 - \tau_{t+1}F_{t+1}$ $- R_{t+1} - \Phi_t - N_t$	0
C_cA_e	$\tau_t F_t$	$\tau_t(1-\rho)F_t$ $-\Delta F_{et}(1-\tau_{t+1})$ $- R_t(1-\rho)G_t$	$\Delta\Phi_t - \tau_t(1-\rho)F_t$ $- R_tH_t*F_a(\Delta T_t)$ $+ R_t(1-\rho)G_t$	$(PF_a)*H_t(\Delta T_t)$ $- R_t(PF_a)*H_t(\Delta T_t)$	$\Delta F_{et} - \Delta\Phi_t + R_tN_t$ $+ \tau_t(1-\rho)F_t$ $- (PF_a)*H_t(\Delta T_t)$	$R_tH_t(\Delta T_t)$ $- \rho\Delta F_{et}(1 - \tau_t)$

The notation here is

$$F_t = 1 - F_{et}$$
$$\rho = \exp(-r_o \Delta T_t)$$

$$G_t = 1 - G_t(\Delta T_t)$$
$$\tau_t = \exp(-r_o T_t)$$

$$F_a = 1 - F_a(\cdot)$$

$$\bar P = 1 - P$$

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